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# An analytical procedure to determine constitutive coefficients of a mixture of two linear elastic solids

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**Abstract**

In this paper, we study the coefficients of constitutive equations of a binary mixture of elastic solids and give an analytical approach to determine them. Assuming a material of two-phase elastic composite with randomly distributed elastic spheres is equivalent to a mixture of two elastic solids, we find the values of unknown coefficients by making use of Boussinesq problem. Furthermore, a mean displacement vector definition is also given.


*Keywords:* Mixture; Composite material; Boussinesq problem; Galerkin vector; Mean displacement vector

## 1. Introduction

The increase in the expectations about the physical and chemical properties of the materials, depending on industrial progress, has caused researches on the multi-component materials to gain increasing importance recently in many technological applications where a single material fails to fulfill our expectations. Much work has been undertaken to find complete systems of equations governing the thermomechanical response of these materials. An acceptable approach based on the ideas and methods of modern continuum mechanics is the theory of mixtures.

Truesdell (1957) was the first to formulate the thermomechanical balance equations for a mixture of general materials. After his pioneering work, a good amount of literature has been generated on the formulation of continuum thermomechanical theories of mixtures. The reader is referred to the works of Bowen

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(1976), Atkin and Craine (1976a,b), Bedford and Drumheller (1983) and Rajagopal and Wineman (1990) regarding the historical development of the theory and detailed analysis of various results.

Using a general thermodynamical theory of interacting continua in the form developed in Green and Naghdi (1965), basic constitutive equations for a mixture of two nonlinear elastic solids have been given by Green and Steel (1966), and later from these general equations, the linearised theory for an isotropic mixture of two elastic solids has been derived by Steel (1967a). In his subsequent study, Steel (1968) stated that the physical motivation for considering such a theory is an attempt to describe the behavior of certain binary alloys or certain kinds of composite material in which each point of the mixture can be considered as being occupied by a particle of each solid when considered on a macroscopic scale.

As is known, in many situations, it might not be possible to experimentally measure the physical quantities associated with each constituent of the mixture. Therefore, we need to seek various methods to determine them. In this study, we shall present a mechanical method to achieve this goal. For this purpose, it is assumed that an isotropic mixture of two linear elastic solids, each having the same constant temperature, is equivalent to a material of two-phase elastic composite with randomly distributed elastic spheres. The correspondence between the actual composite and its equivalent mixture is established on the requirement of equality of the total stresses for the two media.

In the next section, the balance laws and relevant constitutive equations are briefly presented, and then the equations governing the motion of the binary mixture are stated for the case of equilibrium. In the subsequent sections, the displacement vectors and the stress tensors are written in terms of the Galerkin vectors, some relations between the mixture and a linear elastic solid are expressed and Boussinesq problem which is well known problem in classical theory of elasticity is solved for a mixture of two elastic solids. Finally, a comparison between the two solutions is made and unknown constitutive constants are obtained.

## 2. Basic theory

We consider that the mixture of two elastic solids is initially isotropic and the solids have constant densities $\bar{\rho}_1$ and $\bar{\rho}_2$, initially. At an arbitrary time $t$ it is assumed that each point of the mixture is occupied simultaneously by the constituents $C_1$ and $C_2$, which are in motion relative to a fixed system of rectangular Cartesian axes. The positions of the typical particles of the solids at time $t$ are denoted by

$$x_i(t) = x_i(X_1, X_2, X_3, t), \quad y_i(t) = y_i(Y_1, Y_2, Y_3, t), \tag{1}$$

where $X_i$ and $Y_i$ are the reference positions of the particles. These motions are assumed to be one-to-one continuous, and invertible. Since a given position is occupied at time $t$ by a particle of each solid, we may write $x_i(t) = y_i(t)$. All subsequent displacements are assumed to be small, so that we retain only linear terms in the partial stresses and the diffusive force, and we let

$$x_i = X_i + u_i^{(1)}, \quad y_i = Y_i + u_i^{(2)}. \tag{2}$$

Since infinitesimal changes of positions are considered, it may be assumed for convenience that the particles occupy the same position initially, i.e. $X_i = Y_i$.

In order to simplify the problem, thermal effects and exchanges of mass between the constituents are excluded so that the balances of mass for $C_1$ and $C_2$ reduce to

$$\rho_1 = \bar{\rho}_1(1 - e_{mm}), \quad \rho_2 = \bar{\rho}_2(1 - g_{mm}), \tag{3}$$

where $e_{ik}$ and $g_{ik}$ are the strain tensors of elastic solids. In the linear theory they are given by

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial X_k} + \frac{\partial u_k^{(1)}}{\partial X_i}\right), \quad g_{ik} = \frac{1}{2}\left(\frac{\partial u_i^{(2)}}{\partial Y_k} + \frac{\partial u_k^{(2)}}{\partial Y_i}\right). \tag{4}$$

The constitutive equations for the partial stresses and the diffusive force in equilibrium, referred to fixed rectangular Cartesian axes are (Steel, 1967a)

$$\sigma_{ik} = -\alpha_2\delta_{ik} + \lambda_1 e_{mm}\delta_{ik} + 2\mu_1 e_{ik} + \lambda_3 g_{mm}\delta_{ik} + 2\mu_3 g_{ik} - \lambda_5(h_{ik} - h_{ki}), \tag{5}$$

$$\pi_{ik} = \alpha_2\delta_{ik} + \lambda_2 g_{mm}\delta_{ik} + 2\mu_2 g_{ik} + \lambda_4 e_{mm}\delta_{ik} + 2\mu_3 e_{ik} + \lambda_5(h_{ik} - h_{ki}), \tag{6}$$

$$p_k = \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}\frac{\partial g_{mm}}{\partial X_k} + \frac{\bar{\rho}_2\alpha_2}{\bar{\rho}}\frac{\partial e_{mm}}{\partial X_k}, \tag{7}$$

where all of the coefficients are constants, and

$$h_{ik} = \frac{\partial u_k^{(1)}}{\partial X_i} + \frac{\partial u_i^{(2)}}{\partial Y_k}, \quad \bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2, \quad \alpha_2 = \lambda_3 - \lambda_4. \tag{8}$$

The coefficient $\alpha_2$ represents the partial stress in the initial position.

We now assume that instead of the mixture being isotropic as a whole initially, each solid is isotropic i.e. $\lambda_5 = 0$ (Steel, 1967b). If it is also considered that the only interaction terms in the relations (5) and (6) are those involving $\lambda_3$, $\lambda_4$ and $\mu_3$; then $\lambda_1$, $\mu_1$ and $\lambda_2$, $\mu_2$ can be considered as the Lamé elastic constants for solids $C_1$ and $C_2$ respectively when separated (Steel, 1968). The equations of equilibrium are of the forms

$$\sigma_{ik,i} - p_k + F_k = 0, \quad \pi_{ik,i} + p_k + G_k = 0. \tag{9}$$

Here $F_k$ and $G_k$ are the body forces per unit volume acting on the constituents. A comma denotes differentiation with respect to the initial position of each solid. In this paper, we shall restrict our attemption to the absence of body forces. Under these conditions, introducing (5) and (6) into Eq. (9), we have

$$\lambda_1^* e_{mm,k} + \lambda_3^* g_{mm,k} + 2\mu_1 e_{ik,i} + 2\mu_3 g_{ik,i} = 0, \tag{10}$$

$$\lambda_2^* g_{mm,k} + \lambda_4^* e_{mm,k} + 2\mu_2 g_{ik,i} + 2\mu_3 e_{ik,i} = 0, \tag{11}$$

where

$$\lambda_1^* = \lambda_1 - \frac{\bar{\rho}_2\alpha_2}{\bar{\rho}}, \quad \lambda_2^* = \lambda_2 + \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}, \quad \lambda_3^* = \lambda_3 - \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}, \quad \lambda_4^* = \lambda_4 + \frac{\bar{\rho}_2\alpha_2}{\bar{\rho}}. \tag{12}$$

Also, using the relations $(8)_2$ and $(8)_3$ we obtain that $\lambda_3^* = \lambda_4^*$. Substitution of expressions (4) for $e_{ik}$ and $g_{ik}$ into (10) and (11) yield the following equations of equilibrium in vectorial forms:

$$\mu_1\nabla^2\mathbf{u}^{(1)} + \mu_3\nabla^2\mathbf{u}^{(2)} + (\lambda_1^* + \mu_1)\nabla(\nabla\cdot\mathbf{u}^{(1)}) + (\lambda_3^* + \mu_3)\nabla(\nabla\cdot\mathbf{u}^{(2)}) = 0, \tag{13}$$

$$\mu_3\nabla^2\mathbf{u}^{(1)} + \mu_2\nabla^2\mathbf{u}^{(2)} + (\lambda_4^* + \mu_3)\nabla(\nabla\cdot\mathbf{u}^{(1)}) + (\lambda_2^* + \mu_2)\nabla(\nabla\cdot\mathbf{u}^{(2)}) = 0. \tag{14}$$

## 3. The Galerkin vector representation for a binary mixture of elastic solids

Recently, Gürgöze and Dokuz (1999) using the Helmholtz theorem showed that Eqs. (13) and (14) can be reduced to the system of differential equations

$$(\mu_1 + \mu_3)\nabla^2\mathbf{u}^{(1)} + (\lambda_1 + \lambda_4 + \mu_1 + \mu_3)\nabla(\nabla\cdot\mathbf{u}^{(1)}) = 0, \tag{15}$$

$$(\mu_2 + \mu_3)\nabla^2\mathbf{u}^{(2)} + (\lambda_2 + \lambda_3 + \mu_2 + \mu_3)\nabla(\nabla\cdot\mathbf{u}^{(2)}) = 0 \tag{16}$$

under the conditions

$$\mu_1 + \mu_3 \neq 0, \quad \mu_2 + \mu_3 \neq 0, \quad \lambda_1 + \lambda_4 + \mu_1 + \mu_3 \neq 0, \quad \lambda_2 + \lambda_3 + \mu_2 + \mu_3 \neq 0,$$
$$(\lambda_1^* + 2\mu_1)(\lambda_2^* + 2\mu_2) \neq (\lambda_3^* + 2\mu_3)^2, \quad \mu_1\mu_2 \neq \mu_3^2. \tag{17}$$

Since the differential equations (15) and (16) are formally similar to the Navier equations in the classical theory of elasticity, the similar mathematical methods which were essentially developed for a single continuum can be used to solve them. Therefore, in a previous work, Dokuz and Gürgöze (2002) introduced the Galerkin vector definitions $\mathbf{F}_1$ and $\mathbf{F}_2$ to solve the Boussinesq problem as follows:

$$\mathbf{u}^{(1)} = A\nabla^2\mathbf{F_1} - \nabla(\nabla\cdot\mathbf{F_1}), \quad \mathbf{u}^{(2)} = B\nabla^2\mathbf{F_2} - \nabla(\nabla\cdot\mathbf{F_2}), \tag{18}$$

where $A$ and $B$ are given by

$$A = \frac{\mu_1 + \mu_3}{\lambda_1 + \lambda_4 + \mu_1 + \mu_3} + 1, \quad B = \frac{\mu_2 + \mu_3}{\lambda_2 + \lambda_3 + \mu_2 + \mu_3} + 1. \tag{19}$$

In the case of a state of stress possessing axial symmetry the Galerkin vector is replaced by the Love's strain function. This function is defined by a biharmonic equation in the case of zero body forces. Thus, an axisymmetric problem can be solved if we succeed in finding a proper biharmonic function satisfying the boundary conditions. If $Z_1$ and $Z_2$ are two Love's strain functions satisfying equations

$$\nabla^2\nabla^2 Z_1 = 0, \quad \nabla^2\nabla^2 Z_2 = 0 \tag{20}$$

then, the Galerkin vectors become

$$\mathbf{F}_1 = [\,0 \quad 0 \quad Z_1(x,y,z)\,], \quad \mathbf{F}_2 = [\,0 \quad 0 \quad Z_2(x,y,z)\,]. \tag{21}$$

At the conclusion of this section, we write the stress tensors and the diffusive force vector in terms of the Galerkin vectors as follows:

$$\begin{aligned}\boldsymbol{\sigma} = {} & -\alpha_2\mathbf{I} + \lambda_1(A-1)\nabla^2(\nabla\cdot\mathbf{F_1})\mathbf{I} + \mu_1\{A[\nabla(\nabla^2\mathbf{F_1}) + (\nabla^2\mathbf{F_1})\nabla] - 2\nabla\nabla(\nabla\cdot\mathbf{F_1})\} \\ & + \lambda_3(B-1)\nabla^2(\nabla\cdot\mathbf{F_2})\mathbf{I} + \mu_3\{B[\nabla(\nabla^2\mathbf{F_2}) + (\nabla^2\mathbf{F_2})\nabla] - 2\nabla\nabla(\nabla\cdot\mathbf{F_2})\},\end{aligned} \tag{22}$$

$$\begin{aligned}\boldsymbol{\pi} = {} & \alpha_2\mathbf{I} + \lambda_4(A-1)\nabla^2(\nabla\cdot\mathbf{F_1})\mathbf{I} + \mu_3\{A[\nabla(\nabla^2\mathbf{F_1}) + (\nabla^2\mathbf{F_1})\nabla] - 2\nabla\nabla(\nabla\cdot\mathbf{F_1})\} \\ & + \lambda_2(B-1)\nabla^2(\nabla\cdot\mathbf{F_2})\mathbf{I} + \mu_2\{B[\nabla(\nabla^2\mathbf{F_2}) + (\nabla^2\mathbf{F_2})\nabla] - 2\nabla\nabla(\nabla\cdot\mathbf{F_2})\},\end{aligned} \tag{23}$$

$$\mathbf{p} = \alpha_2(A-1)\frac{\bar{\rho}_2}{\bar{\rho}}\nabla[\nabla^2(\nabla\cdot\mathbf{F_1})] + \alpha_2(B-1)\frac{\bar{\rho}_1}{\bar{\rho}}\nabla[\nabla^2(\nabla\cdot\mathbf{F_2})]. \tag{24}$$

## 4. Some deductions

A mixture can be considered as a single continuum with overall Lamé constants $\lambda$ and $\mu$ or overall Young's modulus $E$ and Poisson's ratio $v$. Therefore, we try to make an attempt in this section to obtain some relations between a mixture of two elastic solids and a single elastic continuum. We assume here that the elastic moduli of the solids when mixed are the same as when separate, the only difference when mixed being the addition of interaction terms.

First, we write the total mechanical stress tensor $t_{ik}$ for the mixture defined by

$$t_{ik} = \sigma_{ik} + \pi_{ik}. \tag{25}$$

By substituting Eqs. (5) and (6) into (25), we obtain

$$t_{ik} = [(\lambda_1 + \lambda_4)e_{mm} + (\lambda_2 + \lambda_3)g_{mm}]\delta_{ik} + 2[(\mu_1 + \mu_3)e_{ik} + (\mu_2 + \mu_3)g_{ik}]. \tag{26}$$

It is readily shown that the above equation will have the same form with a single elastic continuum if we define

$$\mu\varepsilon_{ik} = (\mu_1 + \mu_3)e_{ik} + (\mu_2 + \mu_3)g_{ik}, \tag{27}$$

$$\lambda\varepsilon_{mm} = (\lambda_1 + \lambda_4)e_{mm} + (\lambda_2 + \lambda_3)g_{mm}. \tag{28}$$

Thus we have a definition between the partial strains and the mean strain tensor $\varepsilon_{ik}$.

Summing Eq. (9), for $F_k = 0$ and $G_k = 0$, we have

$$\sigma_{ik,i} + \pi_{ik,i} = 0 \tag{29}$$

which is the equation of equilibrium of the mixture. In terms of the displacement vectors, using the definitions (12), it has the form

$$(\mu_1 + \mu_3)\nabla^2\mathbf{u}^{(1)} + (\mu_2 + \mu_3)\nabla^2\mathbf{u}^{(2)} + \nabla[(\lambda_1 + \lambda_4 + \mu_1 + \mu_3)\nabla\cdot\mathbf{u}^{(1)} + (\lambda_2 + \lambda_3 + \mu_2 + \mu_3)\nabla\cdot\mathbf{u}^{(2)}] = 0. \tag{30}$$

Now, let us define the relations

$$\mu\nabla^2\mathbf{w} = (\mu_1 + \mu_3)\nabla^2\mathbf{u}^{(1)} + (\mu_2 + \mu_3)\nabla^2\mathbf{u}^{(2)}, \tag{31}$$

$$(\lambda + \mu)\nabla\cdot\mathbf{w} = (\lambda_1 + \lambda_4 + \mu_1 + \mu_3)\nabla\cdot\mathbf{u}^{(1)} + (\lambda_2 + \lambda_3 + \mu_2 + \mu_3)\nabla\cdot\mathbf{u}^{(2)}, \tag{32}$$

where $\mathbf{w}$ can be called as the mean displacement vector of the mixture. Then (30) will be identical to the Navier's equation of classical theory of elasticity. Integrating Eq. (31), $\mathbf{w}$ is obtained as

$$\mu\mathbf{w} = (\mu_1 + \mu_3)\mathbf{u}^{(1)} + (\mu_2 + \mu_3)\mathbf{u}^{(2)}. \tag{33}$$

Here, it is assumed that $\mathbf{u}^{(1)} = 0$ and $\mathbf{u}^{(2)} = 0$ when $\mathbf{w} = 0$.

It is of course that the strain tensors (27) and (28) can not be independent of the mean displacement vector which is defined in terms of partial displacement vectors as given by Eqs. (32) and (33). The relations between these equations can be seen using the strain tensor of the equivalent single elastic solid. As it is stated in Section 2, in the general theory of mixture, the same final position is occupied by a particle of each solid, that $x_i = y_i$. Since we are considering infinitesimal changes of position, we may write $X_i = Y_i$. Therefore, the strain tensor of the equivalent single elastic solid is

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial w_i}{\partial X_k} + \frac{\partial w_k}{\partial X_i}\right) \tag{34}$$

and, hence, $\varepsilon_{mm} = \nabla\cdot\mathbf{w}$. With the help of (4) and (33) we obtain Eq. (27), and summing (27) and (28) for $i = k = m$ we get Eq. (32).

Finally, in addition to the balance of mass of each constituent, adding Eq. (3) we also have the balance of mass for the mixture as a whole thus (Steel, 1968)

$$\rho = \bar{\rho}(1 - \varepsilon_{mm}), \tag{35}$$

where $\rho = \rho_1 + \rho_2$ and

$$\bar{\rho}\varepsilon_{mm} = \bar{\rho}_1 e_{mm} + \bar{\rho}_2 g_{mm}. \tag{36}$$

In order to test the results, we need to know $\lambda_3$, $\lambda_4$ and $\mu_3$. Generally, we have to determine them from experiments. Since, such experimental data is not available at the present moment, using the solution of Boussinesq problem, we try to make an analytical approach in determination of them in the following sections.

## 5. Boussinesq problem

Let us consider a semi-infinite mixture body under the action of a concentrated normal load $Q$ acting on the boundary plane along the $z$ axis (Fig. 1).

Evidently, this is an axisymmetric problem with the line of action of $Q$ as the axis of symmetry. The symmetry of the problem suggests the use of cylindrical coordinates and Love's strain functions $Z_1 = Z_1\,(r,z)$ and $Z_2 = Z_2(r,z)$.

Previously, Dokuz and Gürgöze (2002) examined this problem for a mixture of two elastic solids as a whole. They used two Love's strain functions and found the unknown quantities with the help of two boundary conditions. As a distinct departure from the preceding work, in this study, we shall employ four Love's strain functions and conditions.

The boundary conditions of the problem for $r > 0$ and $z = 0$ are

$$t_{zz} = (\sigma_{zz} + \alpha_2) + (\pi_{zz} - \alpha_2) = 0, \quad t_{rz} = \sigma_{rz} + \pi_{rz} = 0. \tag{37}$$

Let $\mathbf{Q_1}$ and $\mathbf{Q_2}$ be the surface tractions associated with the solids $C_1$ and $C_2$, respectively, i.e.,

$$(\boldsymbol{\sigma} + \alpha_2\mathbf{I})^{\mathrm{T}}\mathbf{n} = \mathbf{Q_1}, \quad (\boldsymbol{\pi} - \alpha_2\mathbf{I})^{\mathrm{T}}\mathbf{n} = \mathbf{Q_2}, \quad \mathbf{t}^{\mathrm{T}}\mathbf{n} = \mathbf{Q}, \tag{38}$$

where $\mathbf{n}$ is the unit outward normal vector and $\mathbf{Q} = \mathbf{Q_1} + \mathbf{Q_2}$. In this paper, following Tao and Rajagopal (1995), we assume that the boundary surface fractions (and volume fractions) occupied by the solid constituents $C_1$ and $C_2$ are

$$\alpha = \frac{\rho_1}{\bar{\rho}_1} = \frac{\delta V_1}{\delta V} = \frac{\delta S_1}{\delta S}, \quad 1 - \alpha = \frac{\rho_2}{\bar{\rho}_2} = \frac{\delta V_2}{\delta V} = \frac{\delta S_2}{\delta S} \tag{39}$$

and

$$\mathbf{Q_1} = \alpha\mathbf{t}^{\mathrm{T}}\mathbf{n}, \quad \mathbf{Q_2} = (1-\alpha)\mathbf{t}^{\mathrm{T}}\mathbf{n} \tag{40}$$

or

$$Q_1 = \alpha Q, \quad Q_2 = (1-\alpha)Q. \tag{41}$$

Here $\delta V_1$ and $\delta V_2$ denote the volumes occupied by $C_1$ and $C_2$ in an infinitesimal cube of the mixture $\delta V$ at $\mathbf{x}$. Also $\alpha$ is a positive constant subject to the relation $0 < \alpha < 1$. The assumptions (39) are consistent with Mills' volume additivity constraint (Mills, 1967), that is, each constituent in its reference state is assumed to be incompressible and

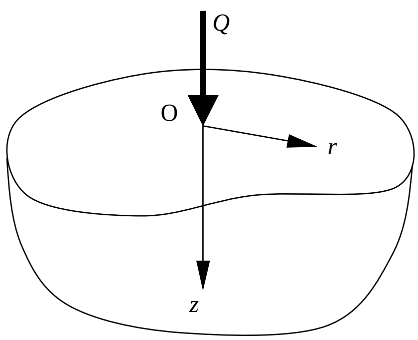


Fig. 1. Boussinesq problem.

$$\frac{\rho_1}{\bar{\rho}_1} + \frac{\rho_2}{\bar{\rho}_2} = 1. \tag{42}$$

Thus, for $r > 0$ and $z = 0$, from Eqs. (37), (38) and (40) we have

$$\sigma_{zz} + \alpha_2 = 0, \quad \pi_{zz} - \alpha_2 = 0, \quad \sigma_{rz} = 0, \quad \pi_{rz} = 0. \tag{43}$$

Now, let us consider a horizontal plane section at a distance $z = h$ from the boundary plane. The normal stresses on this section must be in equilibrium with the surface forces and, consequently, in equilibrium with the loads $Q_1$ and $Q_2$. Therefore, we have the following equilibrium equations in place of the first two boundary conditions in (43) (see Dokuz and Gürgöze, 2002):

$$Q_1 = -\int_0^\infty 2\pi r(\sigma_{zz} + \alpha_2)|_{z=h}\mathrm{d}r, \quad Q_2 = -\int_0^\infty 2\pi r(\pi_{zz} - \alpha_2)|_{z=h}\mathrm{d}r. \tag{44}$$

According to Eq. (20), $Z_1$ and $Z_2$ must be biharmonic functions whose third partial derivatives should define stresses that vanish at infinity. Therefore, we shall seek solutions compatible with Eq. (20), of the following forms:

$$\bar{Z}_1 = N_1 R, \quad \bar{Z}_2 = M_1 R \tag{45}$$

and

$$\bar{\bar{Z}}_1 = N_2[R - z\ln(R + z)], \quad \bar{\bar{Z}}_2 = M_2[R - z\ln(R + z)], \tag{46}$$

where $N_1$, $N_2$, $M_1$ and $M_2$ are constants to be obtained later, and $R = (r^2 + z^2)^{1/2}$. Using the Love's strain functions $\bar{Z}_1, \bar{Z}_2$ and $\bar{\bar{Z}}_1$, $\bar{\bar{Z}}_2$, making some calculations, from (22) and (23) we find the components of stress tensors $\bar{\sigma}_{ik}$, $\bar{\pi}_{ik}$ and $\bar{\bar{\sigma}}_{ik}$, $\bar{\bar{\pi}}_{ik}$, respectively. After superposing these equations, i.e.

$$\sigma_{ik} = \bar{\sigma}_{ik} + \bar{\bar{\sigma}}_{ik}, \quad \pi_{ik} = \bar{\pi}_{ik} + \bar{\bar{\pi}}_{ik} \tag{47}$$

we obtain the components of stress tensors and diffusive force vector as

$$\begin{aligned}\sigma_{rr} = {} & -\alpha_2 + 2\{[(1-A)\lambda_1 + \mu_1]N_1 + [(1-B)\lambda_3 + \mu_3]M_1 + \mu_1 N_2 + \mu_3 M_2\}\frac{z}{r^3} \\ & - 2(\mu_1 N_1 + \mu_3 M_1)\frac{3r^2 z}{R^5} - 2(\mu_1 N_2 + \mu_3 M_2)\frac{1}{R(R+z)},\end{aligned} \tag{48}$$

$$\begin{aligned}\pi_{rr} = {} & \alpha_2 + 2\{[(1-A)\lambda_4 + \mu_3]N_1 + [(1-B)\lambda_2 + \mu_2]M_1 + \mu_3 N_2 + \mu_2 M_2\}\frac{z}{r^3} \\ & - 2(\mu_3 N_1 + \mu_2 M_1)\frac{3r^2 z}{R^5} - 2(\mu_3 N_2 + \mu_2 M_2)\frac{1}{R(R+z)},\end{aligned} \tag{49}$$

$$\sigma_{rz} = 2[(1-A)\mu_1 N_1 + (1-B)\mu_3 M_1 - \mu_1 N_2 - \mu_3 M_2]\frac{r}{R^3} - 2(\mu_1 N_1 + \mu_3 M_1)\frac{3r^2 z}{R^5}, \tag{50}$$

$$\pi_{rz} = 2[(1-A)\mu_3 N_1 + (1-B)\mu_2 M_1 - \mu_3 N_2 - \mu_2 M_2]\frac{r}{R^3} - 2(\mu_3 N_1 + \mu_2 M_1)\frac{3r^2 z}{R^5}, \tag{51}$$

$$\sigma_{\theta\theta} = -\alpha_2 + 2\{[(1-A)\lambda_1 + \mu_1]N_1 + [(1-B)\lambda_3 + \mu_3]M_1\}\frac{z}{R^3} + 2(\mu_1 N_2 + \mu_3 M_2)\frac{1}{R(R+z)}, \tag{52}$$

$$\pi_{\theta\theta} = \alpha_2 + 2\{[(1-A)\lambda_4 + \mu_3]N_1 + [(1-B)\lambda_2 + \mu_2]M_1\}\frac{z}{R^3} + 2(\mu_3 N_2 + \mu_2 M_2)\frac{1}{R(R+z)}, \tag{53}$$

$$\sigma_{zz} = -\alpha_2 + 2\{[(1-A)\lambda_1 + (3-2A)\mu_1]N_1 + [(1-B)\lambda_3 + (3-2B)\mu_3]M_1 - (\mu_1 N_2 + \mu_3 M_2)\}\frac{z}{R^3} - 2(\mu_1 N_1 + \mu_3 M_1)\frac{3z^3}{R^5}, \tag{54}$$

$$\pi_{zz} = \alpha_2 + 2\{[(1-A)\lambda_4 + (3-2A)\mu_3]N_1 + [(1-B)\lambda_2 + (3-2B)\mu_2]M_1 - (\mu_3 N_2 + \mu_2 M_2)\}\frac{z}{R^3} - 2(\mu_3 N_1 + \mu_2 M_1)\frac{3z^3}{R^5}, \tag{55}$$

$$p_r = 6\eta\frac{rz}{R^5}, \quad p_\theta = 0, \quad p_z = -2\eta\frac{r^2 - 2z^2}{R^5}, \tag{56}$$

where

$$\eta = \alpha_2\left[(A-1)N_1\frac{\bar{\rho}_2}{\bar{\rho}} + (B-1)M_1\frac{\bar{\rho}_1}{\bar{\rho}}\right]. \tag{57}$$

The above results can now be substituted Eqs. $(43)_3$, $(43)_4$ and (44) to satisfy the boundary conditions required for Boussinesq's solution. This gives

$$[(A-1)N_1 + N_2]\mu_1 + [(B-1)M_1 + M_2]\mu_3 = 0, \tag{58}$$

$$[(A-1)N_1 + N_2]\mu_3 + [(B-1)M_1 + M_2]\mu_2 = 0, \tag{59}$$

$$Q_1 = 4\pi[(A-1)(\lambda_1 + 2\mu_1)N_1 + (B-1)(\lambda_3 + 2\mu_3)M_1 + \mu_1 N_2 + \mu_3 M_2], \tag{60}$$

$$Q_2 = 4\pi[(A-1)(\lambda_4 + 2\mu_3)N_1 + (B-1)(\lambda_2 + 2\mu_2)M_1 + \mu_3 N_2 + \mu_2 M_2]. \tag{61}$$

Remembering the inequality $\mu_1\mu_2 \neq \mu_3^2$ in (17), from (58) and (59) one gets

$$N_2 = (1-A)N_1, \quad M_2 = (1-B)M_1 \tag{62}$$

and hence, $N_1$ and $M_1$ can be given by the solution of Eqs. (60) and (61) as

$$N_1 = \frac{Q_1(\lambda_2 + \mu_2) - Q_2(\lambda_3 + \mu_3)}{(A-1)D}, \quad M_1 = \frac{Q_2(\lambda_1 + \mu_1) - Q_1(\lambda_4 + \mu_3)}{(B-1)D}, \tag{63}$$

where $D = 4\pi[(\lambda_1 + \mu_1)(\lambda_2 + \mu_2) - (\lambda_3 + \mu_3)(\lambda_4 + \mu_3)]$ and it is assumed that

$$(\lambda_1 + \mu_1)(\lambda_2 + \mu_2) - (\lambda_3 + \mu_3)(\lambda_4 + \mu_3) \neq 0. \tag{64}$$

The displacement vectors of the mixture constituents can be obtained in a manner similar to that used for the stress components (48)–(55). Employing Eqs. (18), (21), (45) and (46), we find the following relations for the displacement components:

$$u_r^{(1)} = \frac{r}{R}\left(\frac{N_1 z}{R^2} + \frac{N_2}{R+z}\right), \quad u_z^{(1)} = \frac{1}{R}\left(AN_1 + \frac{N_1 z^2}{R^2}\right), \tag{65}$$

$$u_r^{(2)} = \frac{r}{R}\left(\frac{M_1 z}{R^2} + \frac{M_2}{R+z}\right), \quad u_z^{(2)} = \frac{1}{R}\left(BM_1 + \frac{M_1 z^2}{R^2}\right). \tag{66}$$

## 6. The determination of $\lambda_3$, $\lambda_4$ and $\mu_3$

As indicated before, a mixture can be considered to be a single continuum with overall elastic constants $\lambda$ and $\mu$ or $E$ and $v$. In that case, we need to know the solution of Boussinesq problem for single continua. The solution for a linear elastic solid is given by the following equations (Fung, 1968):

$$w_r = \frac{(1+v)Q}{2\pi ER}\left[\frac{rz}{R^2} - \frac{(1-2v)r}{R+z}\right], \quad w_z = \frac{(1+v)Q}{2\pi ER}\left[2(1-v) + \frac{z^2}{R^2}\right], \tag{67}$$

$$t_{rr} = \frac{Q}{2\pi R^2}\left[\frac{(1-2v)R}{R+z} - \frac{3r^2z}{R^3}\right], \quad t_{\theta\theta} = \frac{(1-2v)Q}{2\pi R^2}\left[\frac{z}{R} - \frac{R}{R+z}\right], \tag{68}$$

$$t_{zz} = -\frac{3Qz^3}{2\pi R^5}, \quad t_{rz} = -\frac{3Qrz^2}{2\pi R^5}. \tag{69}$$

Since we consider the mixture as an elastic solid, the components of total stress tensor $\sigma_{ik} + \pi_{ik}$ and mean displacement vector $w_k$ must be equal to above relevant equations. By comparing Eqs. (48)–(55), (68) and (69) for $\sigma_{ik} + \pi_{ik}$, and (65)–(67) using (33), we find two linear independent equations

$$(A-1)(\mu_1 + \mu_3)N_1 + (B-1)(\mu_2 + \mu_3)M_1 = \frac{Q(1-2v)}{4\pi}, \tag{70}$$

$$(A-1)(\lambda_1 + \lambda_4)N_1 + (B-1)(\lambda_2 + \lambda_3)M_1 = \frac{Qv}{2\pi} \tag{71}$$

for three unknowns $\lambda_3$, $\lambda_4$ and $\mu_3$. An additional equation can be obtained, by comparison between (32) and (36), as follows:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + 2\mu_3 = \lambda + \mu, \tag{72}$$

where Eq. $(8)_2$ is used. Also, with the help of $(8)_2$, we define a new quantity $\beta$ $(0 < \beta < 1)$ to which we shall need later as

$$\beta = \frac{\bar{\rho}_1}{\bar{\rho}} = \frac{\lambda_1 + \lambda_4 + \mu_1 + \mu_3}{\lambda + \mu}, \quad 1 - \beta = \frac{\bar{\rho}_2}{\bar{\rho}} = \frac{\lambda_2 + \lambda_3 + \mu_2 + \mu_3}{\lambda + \mu}. \tag{73}$$

As an example the experimental data for the materials of two-phase elastic composites with randomly distributed elastic spheres, recorded by Smith (1976), are used here. The material properties involved in that experiment are:

$$E^* = 76 \text{ GPa}, \quad v^* = 0.23, \quad E' = 3 \text{ GPa}, \quad v' = 0.4. \tag{74}$$

Here, $E^*$, $v^*$ and $E'$, $v'$ are the material constants of the spherical inhomogenities and the matrix, respectively. The relationships between $\lambda$, $\mu$ and $E$, $v$ are given by

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)}. \tag{75}$$

Hence, we assume for the spherical inhomogenities

$$\lambda_1 = 26.32 \text{ GPa}, \quad \mu_1 = 30.89 \text{ GPa} \tag{76}$$

and for the matrix

$$\lambda_2 = 4.29 \text{ GPa}, \quad \mu_2 = 1.07 \text{ GPa}. \tag{77}$$

Table 1
The experimental data of composite

| $\alpha$ | $E$ (GPa) | $\mu$ (GPa) |
|---|---|---|
| 0.1 | 3.75 | 1.33928571 |
| 0.225 | 5.1 | 1.82142857 |
| 0.3 | 6 | 2.14285714 |
| 0.398 | 7.9 | 2.86071429 |
| 0.495 | 12.1 | 4.41428571 |

Some experimental data by Smith (1976) are listed in Table 1, in which $\alpha$ is the volume fraction of the spherical inhomogenities, $E$ is the overall Young's modulus and $\mu$ is the overall shear modulus.

By using the data listed in (76), (77) and Table 1, the coefficients $\lambda_3$, $\lambda_4$ and $\mu_3$ can be determined from Eqs. (70)–(72). But (70) and (71) are nonlinear algebraic equations, that is, we have a number of solutions for $\lambda_3$, $\lambda_4$ and $\mu_3$. Therefore, bearing in mind that $0 < \beta < 1$, we must test the results with the conditions given by Borrelli and Patria (1983), and with the inequalities (17) and (64). For an isotropic mixture Borrelli and Patria (1983) recorded that ($\lambda_5 = 0$)

$$\begin{aligned}&\mu_1 \geqslant 0, \quad \mu_2 \geqslant 0, \quad \lambda_1 + 2\mu_1 - (1-\beta)\alpha_2 \geqslant 0, \quad \lambda_2 + 2\mu_2\beta\alpha_2 \geqslant 0, \quad \mu_3^2 \leqslant \mu_1\mu_2,\\ &(\lambda_3 + 2\mu_3 - \beta\alpha_2)^2 \leqslant [\lambda_1 + 2\mu_1 - (1-\beta)\alpha_2][\lambda_2 + 2\mu_2 + \beta\alpha_2].\end{aligned} \tag{78}$$

In addition, we assume that for very small variations of $\alpha$ the elastic constants of composite in Table 1 do not change. For instance, $E$ and $\mu$ are the same for $\alpha = 0.22499999$, $\alpha = 0.225$ and $\alpha = 0.22500001$. Under these conditions some calculated results and plots for $\lambda_3$, $\lambda_4$ and $\mu_3$ are given in Table 2 and Figs. 2–4.

Taking, for example, $\alpha = 0.299999997 (\approx 0.3)$ and the relevant values of coefficients in Tables 1 and 2 ($E = 6$, $\mu = 2.14285714$, $v = 0.4$, $\lambda = 8.57142867$, $\lambda_3 = 0.09026783$, $\lambda_4 = -48.54958352$, $\mu_3 = -1.6981993$, $\beta = 0.649806937$) the displacements vectors, stress tensors and diffusive force vector are obtained for the point $r = 0.02$ and $z = 0.03$ as

$$\mathbf{u}^{(1)} = [-0.00514759Q \quad 0 \quad 0.0374949Q], \quad \mathbf{u}^{(2)} = [-1.648Q \quad 0 \quad -4.90598Q], \tag{79}$$

$$\boldsymbol{\sigma} + \alpha_2\mathbf{I} = [\{-13.176Q \quad 0 \quad -361.73Q\} \quad \{0 \quad 229.267Q \quad 0\} \quad \{-361.73Q \quad 0 \quad -311.905Q\}], \tag{80}$$

$$\boldsymbol{\pi} - \alpha_2\mathbf{I} = [\{-67.4885Q \quad 0 \quad 220.686Q\}\{0 \quad -222.259Q \quad 0\}\{220.686Q \quad 0 \quad 100.339Q\}], \tag{81}$$

$$\mathbf{p} = [37236.1Q \quad 0 \quad 28961.4Q], \tag{82}$$

where $\alpha_2 = 48.6399$.

Table 2
Some calculated results for $\lambda_3$, $\lambda_4$ and $\mu_3$

| $\alpha$ | $\beta$ | $\lambda_3$ (GPa) | $\lambda_4$ (GPa) | $\mu_3$ (GPa) |
|---|---|---|---|---|
| 0.099999991 | 0.649916722 | −0.821656176 | −50.66384294 | −2.194036158 |
| 0.225000007 | 0.649869671 | −0.300049793 | −49.42028078 | −1.871263283 |
| 0.299999997 | 0.649806937 | 0.090267830 | −48.54958352 | −1.698199300 |
| 0.398000005 | 0.649914470 | 0.348816726 | −47.90412740 | −1.508831199 |
| 0.495000008 | 0.650533125 | 1.364605879 | −45.35216080 | −0.766133255 |

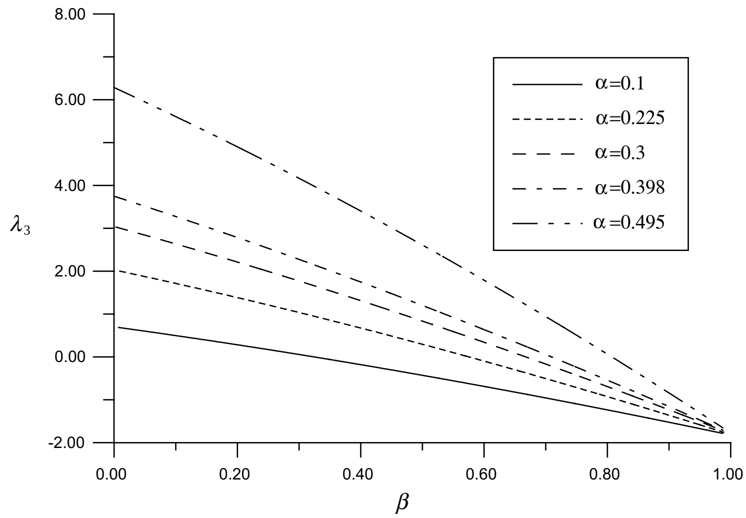


Fig. 2. The variation of $\lambda_3$ with $\beta$.

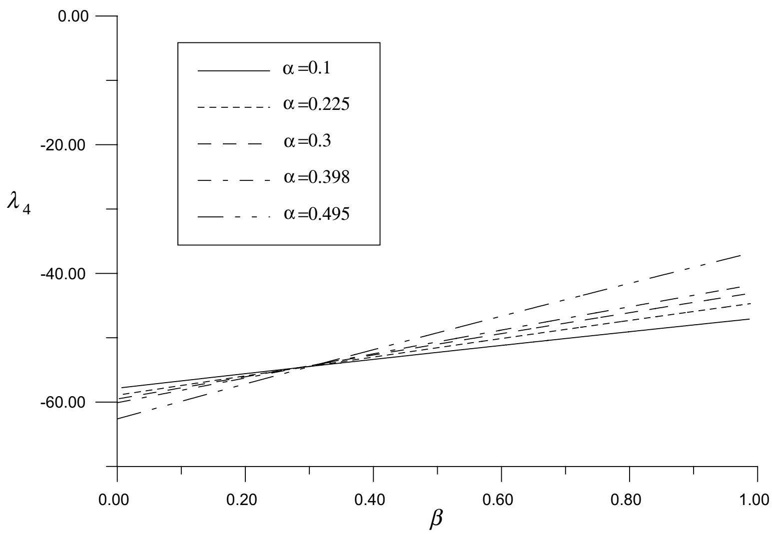


Fig. 3. The variation of $\lambda_4$ with $\beta$.

On the other hand, for a single elastic solid, from (67)–(69) we find

$$\mathbf{w} = [0.413001Q \quad 0 \quad 1.94902Q], \tag{83}$$

$$\mathbf{t} = [\{-80.6645Q \quad 0 \quad -141.044Q\}\{0 \quad 7.00805Q \quad 0\}\{-141.044Q \quad 0 \quad -211.566Q\}]. \tag{84}$$

By employing Eq. (33), the same result in Eq. (83) can be obtained. Summation of (80) and (81) give the total stress tensor $\mathbf{t}$ in Eq. (84). Furthermore, Eqs. (27), (28) and (32) are also confirmed by using these values.

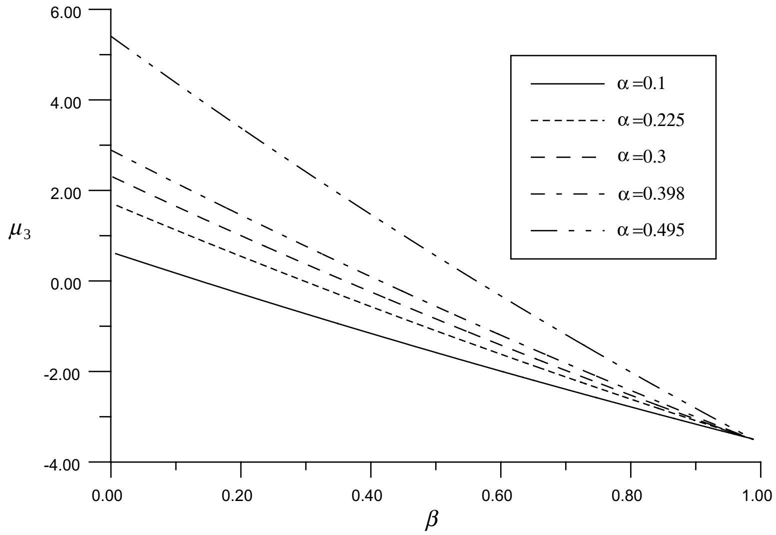


Fig. 4. The variation of $\mu_3$ with $\beta$.

## 7. Concluding remarks

In order to compare theoretical solutions with experimental results, it is necessary to know all response functions in the constitutive equations of the mixture. However, the determination of these functions for a mixture is much more difficult than that for a single continuum, owing to a large number of response functions appearing in the constitutive equations. The presented method is an alternative attempt to obtain some information about the response functions of the mixture. In this work, we have followed an analytical way to determine them.

In this study, a linear elastic isotropic mixture of two elastic solids has been taken into account. The governing equations are formulated in terms of the displacement vectors. These equations have been solved for Boussinesq problem by using the Galerkin vectors. Then, the results have been compared with the existing analytical solution of a single elastic solid and it has been seen that, there are three algebraic equations to determine three unknown constants. Additionally, in this paper, a mean displacement vector definition has also been given. When this definition is compared with the known solution of classical theory of elasticity, a certain agreement is shown between both of them.

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